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On the 1-Cohomology of Finite Groups of Lie Type

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For each simply connected semisimple algebraic group G defined and split over the prime field $\mathbb{F}_p$, we establish a uniform bound on n above which all of the first cohomology groups with values in the simple modules for the finite group $G(n)$ are determined by those for the algebraic group $G(\tilde{\mathbb{F}}_p)$. © 1997 Academic Press

INTRODUCTION

Let G be a simply connected semisimple algebraic group defined and split over the prime field $\mathbb{F}_p$. Write $G(n)$ for the finite subgroup of rational points over the field with p^n elements. For each choice of algebraic group G and prime p, we shall describe an explicit finite set of weights Ξ so that if $n \in \mathbb{N}$ and M is a simple $G(n)$-module with $H^1(G(n), M) \neq 0$, then M is up to Frobenius twist isomorphic to the restriction to $G(n)$ of an irreducible G-module with highest weight in Ξ. The remarkable fact is that Ξ does not depend on n, despite the fact that the number of irreducibles for $G(n)$ grows with n. Thus the task of computing all $H^1(G(n), M)$ for simple M involves only finitely many M. For fixed M the existence of "generic cohomology" [4] allows one to restrict to an explicit finite set of n. Thus only finitely many cases are left for each G, p. The proof of this result also goes through for twisted groups that result from automorphisms of the Dynkin diagram for G.

It has long been known that there is a relationship between cohomology for the algebraic groups and cohomology for the finite Chevalley groups.

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In [4], Cline, Parshall, Scott, and van der Kallen showed that for each *fixed* highest weight λ, there is a natural number N such that $n > N$ implies that the restriction map

$$H^1(G, L(\lambda)_e) \to H^1(G(n), L(\lambda)_e)$$

is an isomorphism for sufficiently large e, where V_e denotes the eth Frobenius twist of the module V. Avrunin [3] has extended this result to include the twisted groups. Since cohomology is always invariant under Frobenius twisting for the finite groups and in most cases for the algebraic groups, we obtain isomorphisms

$$H^1(G(n), L(\lambda)) \cong H^1(G, L(\lambda))$$

for all $n > N$ if G is a simply connected semisimple algebraic group not of type C_l if $p = 2$. (In the latter case, we have

$$H^1(G(n), L(\lambda)) \cong H^1(G, L(\lambda)_1)$$

for all $n > N$.) Of course, N depends on λ, and generally increases with the height of λ.

The goal of this paper is thus to reverse the order of the quantifiers in the above statement, and for each simply connected semisimple algebraic group, establish a uniform bound N so that if $n > N$, then the above isomorphisms exist for *all* p^n-restricted weights λ. In other words, if $n > N$, all of the first cohomology groups (with values in the simple modules) for the finite group $G(n)$ are known in terms of the first cohomology groups for the algebraic group (and the latter are generally easier to compute.) Thus, the infinite problem of determining all of the first cohomology groups for all of the finite groups $G(n)$ is reduced to the finite problem of determining them for $n \leq N$. The bulk of the argument involves the reduction of the problem to a reasonable finite number of cases where the cohomology might be nonzero. We show that the 1-cohomology groups vanish in a large number of cases by using a generalization of the induction step that Alperin used in [1], obtained from the long exact sequence in cohomology.

1. NOTATION AND PRELIMINARIES

We consider the problem of computing the first cohomology groups with values in the simple modules over fields of the natural characteristic for the groups of Lie type in characteristic p. Fix an algebraic closure F of $\mathbb{F}_p$, and regard finite extensions of $\mathbb{F}_p$ as subfields of F. We denote by G a simply connected semisimple algebraic group and by $G(n)$ the corresponding finite group, for $n \in \mathbb{N}$. Let T be a maximal torus of G, and for dominant weights $\mu \in X^+(T)$, with respect to a fixed choice of Borel subgroup containing T, let $L(\mu)$ denote the unique (up to isomorphism) simple module for G with highest weight μ. We will denote coroots using the standard "check" notation (e.g., $\check{\alpha}$) and by ρ and $\check{\rho}$, we shall mean the half-sums of the positive roots and coroots, respectively. For a module M, over G or $G(n)$, we denote by M^* its dual (contragredient). We denote by M_i, or occasionally by $M^{(p^i)}$, the ith Frobenius twist of M. The set of (isomorphism classes of) simple modules for $G(n)$ is comprised of the restriction to $G(n)$ of the p^n-restricted modules for G. By Steinberg's tensor product theorem, this will be exactly the restriction to $G(n)$ of the set of modules of the form

$$\begin{aligned} L(\mu_0) \otimes L(\mu_1)_1 \otimes \cdots \otimes L(\mu_n)_n \\ \cong L(\mu_0) \otimes L(p\mu_1) \otimes \cdots \otimes L(p^n\mu_n) \\ \cong L(\mu_0 + p\mu_1 + \cdots + p^n\mu_n), \end{aligned}$$

as $\mu_0, \ldots, \mu_n$ range over the restricted weights (i.e., those integral weights λ for which $0 \leq \langle \lambda, \check{\alpha}_i \rangle < p$ for each simple root α_i).

For a finite set I of natural numbers, and a restricted simple module V, we let $V_I = \bigotimes_{i \in I} V_i$. Let S denote the restricted Steinberg module (i.e., $S = St_1 = L((p-1)\rho)$). The collection of simple $FG(n)$-modules then consists of the set of all (isomorphism classes of) modules of the form

$$S_I \otimes \bigotimes_{i \in J} X(i)_i,$$

where I, J are disjoint subsets of $N = \{0, 1, \ldots, n-1\}$, and $X(i)$ are restricted simple modules other than S or F. (An empty tensor product denotes F.) It is well known that the module S_N is projective; it is the Steinberg module for $G(n)$. The group of field automorphisms $\mathrm{Gal}(F_{p^n}/F_p)$ acts on the set of isomorphism classes of simple $FG(n)$-modules by acting on the set of ordered $(p^r - 1)$-tuples of disjoint subsets of N, where r is the rank of the root system. The automorphism $\gamma \mapsto \gamma^{p^i}$ acts by adding i to each element of N and taking the remainder modulo n. Thus, the main result of the paper can be stated as follows:

THEOREM. *Let p be any prime. Let G be a simply connected semisimple algebraic group defined and split over the prime field $\mathbb{F}_p$. Let N_0 be defined by*

$$N_0 = \max_i \left\{ \left[\log_p \left(\frac{2\langle \check{\rho}, (p-1)\rho \rangle}{\langle \check{\rho}, \lambda_i \rangle} \right) \right] + 1 \right\}$$

(*where the* λ_i *denote the fundamental dominant weights, where* $\rho, \check{\rho}$ *denote the half-sums of the positive roots and coroots, respectively, and where* [ ] *denotes the greatest integer function.*)

If $n \geq N_0$, *then*

$$H^1\Big(G(n), S_I \otimes \bigotimes_{i \in J} X(i)_i\Big) \cong 0$$

(*where* I, J *denote disjoint subsets of* $\{0, 1, \ldots, n-1\}$ *and the* $X(i)$ *are restricted simple modules* $\notin \{F, S\}$ *for each* $i \in J$) *whenever* J *is not Galois conjugate to a subset of*

$$\mathcal{N}_0 = \{0, 1, 2, \ldots, N_0 - 1\},$$

whenever $|I| > 1$, *or whenever both* I *and* J *are nonempty. This statement also holds when* $G(n)$ *is replaced by* G, *if* I, J *are allowed to be disjoint finite sets of nonnegative integers and conjugation is by Frobenius twisting.*

The result for G follows from the result for $G(n)$ because of Theorem 7.1 of [4], which asserts that the restriction map

$$\mathrm{Ext}^1_G(F, L(\lambda)_e) \to \mathrm{Ext}^1_{FG(n)}(F, L(\lambda)_e)$$

is injective if λ is p^n-restricted, and that it is an isomorphism for sufficiently large e if n is larger than a bound which depends on λ. This fact yields the following result for algebraic groups (by considering the five term sequence from the Lyndon–Hochschild–Serre spectral sequence for the pair (G, G_1)).

COROLLARY 1. *For each prime* p, *and each simply connected semisimple algebraic group* G *defined and split over the prime field* $\mathbb{F}_p$, *let* N_0 *be defined as in the main theorem. Then for each restricted weight* $\mu \in X_1(T)$, *the socle of the* G*-module*

$$H^1(G_1, L(\mu))$$

is p^{N_0}*-restricted.*

Once N_0 is determined, Cline, Parshall, Scott, and Van der Kallen's result on generic cohomology immediately yields the following:

COROLLARY 2. *For each prime* p, *and each simply connected semisimple algebraic group* G (*not of type* C_l *if* $p = 2$) *defined and split over the prime field* $\mathbb{F}_p$, *there exists a natural number* $N(G, p)$ *such that* $n > N(G, p)$ *implies that*

$$H^1(G(n), L(\lambda)) \cong H^1(G, L(\lambda))$$

for every p^n*-restricted weight* λ. (*In the exceptional case, we have*

$$H^1(G(n), L(\lambda)) \cong H^1(G, L(\lambda)_1)$$

for every p^n*-restricted weight* λ *if* $n > N(G, p)$.)

The number $N(G, p)$ mentioned in Corollary 2 can be written down explicitly via Cline, Parshall, Scott, and Van der Kallen's formulas in [4] (by considering the maximum over all of the p^{N_0}-restricted weights),

$$N(G, p) = \max_k \left(\left[\log_p \left\{ t \cdot (p^{N_0} - 1) \cdot \langle \rho, \check{\lambda}_k \rangle + 1 \right\} \right] + \left[\frac{ct - 1}{p - 1} \right] + 2 \right),$$

where t is the torsion coefficient of the fundamental group, c is the maximal coefficient of the simple roots in the expression for the highest root, and [ ] denotes the greatest integer function. (Here, the $\check{\lambda}_k$ are the fundamental dominant weights of the dual root system, i.e., $\langle \alpha_j, \check{\lambda}_k \rangle = \delta_{jk}$.) An analogous theorem is obtained for twisted groups, if $G(n)$ is replaced by $G_\pi(n)$ where π is an automorphism of the Dynkin diagram for G and thus the group automorphism of G induced by π composed with the nth Frobenius endomorphism of G is a surjective rational endomorphism of G having a finite fixed-point set $G_\pi(n)$. In the case of the twisted groups, the formula for N_0 is the same, but the computation of N is slightly more complicated (cf. [3]).

2. GENERALIZATION OF MODULE "LENGTH"

Most of our results will hinge on whether or not particular simple modules appear as composition factors of certain tensor products of simple modules; the main tool for this type of analysis will be the concept of module "mass," as first introduced in the papers of Sin [7–9]. It is an analogue of the *length* of a module as defined by Harish-Chandra [5] and is a useful inductive invariant.

We first define "mass" for modules over the algebraic group, then extend the definition to modules over the finite subgroups. In the following lemmas, we let G be an arbitrary semisimple, simply connected, algebraic group over an algebraically closed field of characteristic p. Let $\mathbf{E} = \mathbf{E}_{\mathbb{R}} = X(T) \otimes_{\mathbb{Z}} \mathbb{R}$. Fix some $f \in \mathbf{E}^*$ such that $f(\alpha_i) > 0$ for all $\alpha_i \in \Delta$. For example, we may take $f = (t\check{\rho}, \cdot\,)$, where $\check{\rho} = 1/2\sum_{\alpha \in \Phi^+} \check{\alpha}$, and where t is the torsion coefficient of $X(T)/\mathbb{Z}\Phi$; this will ensure that "mass" will take values in $\mathbb{Z}^+$. This choice of f will also result in "mass" being invariant under the automorphisms of the weight lattice induced by graph automorphisms of the root lattice. Define the (p-restricted) "mass" of a module, $m(V) \in \mathbb{R}$, for G-modules V as follows:

(i) For $\lambda = \sum_{i=0}^{r} p^i \lambda_i \in X(T)^+$ (where $\lambda_i \in X_1(T)$ for all i), we let

$$m(\lambda) = \sum_{i=0}^{r} f(\lambda_i).$$

(ii) Define

$$m(V) = \sup\{m(\lambda) : L(\lambda) \text{ is a composition factor of } V\}.$$

(In particular, we have $m(L(\lambda)) = m(\lambda)$.)

We may also define, for any $k \in \mathbb{N}$, the p^k-restricted mass, by letting $m_{p^k}(\lambda) = \sum_{i=0}^{r} p^{r(i,k)} f(\lambda_i)$ (and extending to nonsimple modules as in (ii) above) where $r(i, k)$ is the least nonnegative residue of i mod k. Whenever we refer to "mass" without an adjective, we shall mean p-restricted mass. There is a natural way of extending the definition of p^k-restricted mass to $G(n)$-modules by representing the simple modules as restrictions to $G(n)$ of G-modules with p^n-restricted highest weight; it can then be shown that the p^k-restricted mass of a G-module is $\geq$ to the p^k-restricted mass (as $G(n)$-module) of its restriction to $G(n)$. The definition of p^k-restricted mass can similarly be extended to $G_\pi(n)$ modules. We will have occasion to use only p-restricted mass and p^n-restricted mass. We often utilize the fact that p-restricted mass is invariant under twisting. We are careful to ensure that (p-restricted) mass is invariant under the automorphisms of the weight lattice induced by graph automorphisms of the root lattice, so that the invariance will still hold for the twisted groups.

LEMMA 2.1. *Let $\lambda, \lambda' \in X(T)^+$, with $\lambda = \sum_{i=0}^{r} p^{ki}\lambda_i$, $\lambda' = \sum_{i=0}^{r} p^{ki}\lambda'_i$ (where $\lambda_i, \lambda'_i \in X_k(T)$ for all i). Then $m_{p^k}(L(\lambda) \otimes L(\lambda')) \leq m_{p^k}(\lambda) + m_{p^k}(\lambda')$ with equality if and only if $\lambda_i + \lambda'_i \in X_k(T)$ for all i, in which case $L(\lambda + \lambda')$ is the unique composition factor of $L(\lambda) \otimes L(\lambda')$ of greatest p^k-restricted mass.*

Proof. *Case* 1. λ, λ' both p^k-restricted.

Suppose $V = L(\nu)$ is a composition factor of $L(\lambda) \otimes L(\lambda')$. Then $\nu \preccurlyeq \lambda + \lambda'$ in the $\mathbb{Z}^+\Delta$ (usual) partial order. If $\nu = \sum p^{ki}\nu_i$ ($\nu_i \in X_k(T)$), then we have

$$m_{p^k}(\nu) = \sum f(\nu_i) \leq \sum p^{ki} f(\nu_i) = f(\nu) \leq f(\lambda + \lambda')$$
$$= f(\lambda) + f(\lambda') = m_{p^k}(\lambda) + m_{p^k}(\lambda')$$

with equality if and only if $\nu = \nu_0 \in X_k(T)$ and $\nu = \lambda + \lambda'$. Thus,

$$m_{p^k}(L(\lambda) \otimes L(\lambda')) \leq m_{p^k}(\lambda) + m_{p^k}(\lambda')$$

with equality if and only if $\nu = \lambda + \lambda' \in X_k(T)$.

Case 2. $\{\lambda, \lambda'\} \not\subseteq X_k(T)$.

We induct on the quantity $m_{p^k}(\lambda) + m_{p^k}(\lambda')$. Write $\lambda = \lambda_0 + p^k\bar{\lambda}$, $\lambda' = \lambda'_0 + p^k\bar{\lambda}'$. Since p^k-restricted mass is preserved under untwisting by powers of p^k, we may assume that $\lambda_0 + \lambda'_0 \neq 0$. Also, we have $\bar{\lambda} + \bar{\lambda}' \neq 0$ by assumption. Now,

$$\begin{aligned} m_{p^k}(L(\lambda) \otimes L(\lambda')) &= m_{p^k}\left(L(\lambda_0) \otimes L(\lambda'_0) \otimes L\left(p^k\overline{\lambda}\right) \otimes L\left(p^k\overline{\lambda}'\right)\right) \\ &= m_{p^k}(L(\nu) \otimes L(\nu')) \end{aligned}$$

for some composition factors $L(\nu), L(\nu')$ of $L(\lambda_0) \otimes L(\lambda'_0), L(p^k\overline{\lambda}) \otimes L(p^k\overline{\lambda}')$, respectively. By induction then,

$$m_{p^k}(\nu) \leq m_{p^k}(\lambda_0) + m_{p^k}(\lambda'_0)$$

and

$$m_{p^k}(\nu') \leq m_{p^k}\left(p^k\overline{\lambda}\right) + m_{p^k}\left(p^k\overline{\lambda}'\right).$$

If equality holds in both, we would have that (again by induction) $\lambda_i + \lambda'_i \in X_k(T)$ for all i, and that $L(\nu) = L(\lambda_0 + \lambda'_0)$ and $L(\nu') = L(p^k\overline{\lambda} + p^k\overline{\lambda}')$ are the unique composition factors of greatest p^k-restricted mass of $L(\lambda_0) \otimes L(\lambda'_0)$ and $L(p^k\overline{\lambda}) \otimes L(p^k\overline{\lambda}')$, respectively. Thus, $L(\lambda + \lambda') = L(\nu) \otimes L(\nu')$ would be the unique composition factor of $L(\lambda) \otimes L(\lambda')$ of greatest p^k-restricted mass $m_{p^k}(\lambda + \lambda') = m_{p^k}(\lambda) + m_{p^k}(\lambda')$. Otherwise, we would have $m_{p^k}(\nu) + m_{p^k}(\nu') < m_{p^k}(\lambda) + m_{p^k}(\lambda')$, so that the induction hypothesis could be applied to $L(\nu) \otimes L(\nu')$ to conclude that

$$\begin{aligned} m_{p^k}(L(\lambda) \otimes L(\lambda')) &= m_{p^k}(L(\nu) \otimes L(\nu')) \\ &\leq m_{p^k}(\nu) + m_{p^k}(\nu') < m_{p^k}(\lambda) + m_{p^k}(\lambda'). \qquad \blacksquare \end{aligned}$$

In the following corollary, define $\theta = \min_{\beta \in X_1(T) \setminus \{0\}} m(\beta)$. This quantity will be used frequently throughout arguments involving module mass.

Corollary 2.2. *If $\lambda, \lambda' \in X_k(T)$ and $L(\nu)$ is a composition factor of $L(\lambda) \otimes L(\lambda')$ with $\nu \notin X_k(T)$, then $m_{p^k}(L(\nu)) \leq m_{p^k}(\lambda) + m_{p^k}(\lambda') - (p^k - 1) \cdot \theta$.*

Proof. Suppose $\nu = \sum_{i=0}^r p^{ki}\nu_i$ is the p^k-adic expansion of ν. We rewrite the inequality from the proof of Case 1 of Lemma 2.1,

$$\begin{aligned} &m_{p^k}(\lambda) + m_{p^k}(\lambda') - m_{p^k}(\nu) \\ &\quad = f(\lambda + \lambda') - \sum f(\nu_i) \\ &\quad \geq f(\nu) - \sum f(\nu_i) = \sum p^{ki} f(\nu_i) - \sum f(\nu_i) \\ &\quad = \sum (p^{ki} - 1) m_{p^k}(\nu_i) \geq (p^k - 1) m_{p^k}(\nu_j) \end{aligned}$$

for some $1 \leq j \leq r$ with $\nu_j \neq 0$, by assumption on ν. $\blacksquare$

We can now elaborate on the significance of the quantity N_0 of the main theorem of the paper. The definition of N_0 simply ensures that $N_0 \in \mathbb{N}$ with

$$p^{N_0}\theta > 2m(S),$$

i.e.,

$$N_0 > \log_p\left(\frac{2m(S)}{\theta}\right),$$

where $\theta = \min_{\beta \in X_1(T) \setminus \{0\}} m(\beta)$. (We observe for future reference that the quantity θ defined above also satisfies the equality

$$\theta = \min_{\beta \in X_1(T) \setminus \{(p-1)\rho\}} (m(S) - m(\beta)),$$

because of the way mass was defined using a linear functional.) Note that $N_0 \geq 2$, since $S = L((p-1)\rho)$. We shall assume from now on that $n \geq N_0$. Thus, all modules of the form $A \otimes B$, where A and B are simple restricted, will be p^{N_0}- (and hence p^n-) restricted.

The quantity N_0 also makes an appearance in the following important lemma:

LEMMA 2.3. *For $i = 0, 1, \ldots, n-1$, let $X(i)$ and $Y(i)$ be restricted simple modules, with $m(X(i)) \geq m(Y(i))$. Let Q be any $G(n)$-module. Then*

$$m_{p^n}\left(\bigotimes_{i=0}^{n-1} X(i)_i\right) > m_{p^n}\left(Q \otimes \bigotimes_{i=0}^{n-1} Y(i)_i\right)$$

(a) *if $m_{p^n}(Q) \leq p \cdot m(S)$ and there exist distinct i, j with $j \geq 1$, such that $m(X(i)) > m(Y(i))$, $X(j) = S$, and $Y(j) = F$, or*

(b) *if $m_{p^n}(Q) < p \cdot m(S)$ and there exists $j \geq 1$ with $X(j) = S$, $Y(j) = F$, or*

(c) *if $m_{p^n}(Q) \leq 2m(S)$ and there exist $i \geq N_0$ with $m(X(i)) \geq m(Y(i)) + \theta$ (e.g., if $X(i) = S \neq Y(i)$.)*

Proof. The result follows immediately in cases (a) and (b) by the inequality of Lemma 2.1. The proof of case (c) requires only Lemma 2.1 and the definition of N_0. ∎

3. REDUCTION OF THE PROBLEM

We show that the 1-cohomology groups for the finite groups vanish in a large number of cases. The following lemma is a generalization of Alperin's induction step [1].

LEMMA 3.1. *Let D be any $FG(n)$-module, let A, B be simple $FG(n)$-modules, and let E be any nonzero quotient of $B \otimes D$. Let $Z(A, B)$ denote the (unique up to isomorphism) $FG(n)$-module with head isomorphic to A, and radical isomorphic to a direct sum of $d = \dim_F(\mathrm{Ext}^1_{FG(n)}(A, B))$ copies of B. Then surjectivity of the natural map*

$$\mathrm{Hom}_{FG(n)}(A \otimes D, E) \to \mathrm{Hom}_{FG(n)}(Z(A, B) \otimes D, E)$$

implies that $\dim_F(\mathrm{Ext}^1_{FG(n)}(A, B)) \leq \dim_F(\mathrm{Ext}^1_{FG(n)}(A \otimes D, E))$.

The proof of Lemma 3.1 is an easy exercise easy exercise in applying the long exact sequence in cohomology for $FG(n)$. In our applications of Lemma 3.1, we will prove surjectivity by showing $\mathrm{Hom}_{FG(n)}(Z(A, B) \otimes D, E) = 0$. In most cases we can simply check that A is not a composition factor of $D^* \otimes E$.

We define the diameter of a subset I of N as

$$\mathrm{diam}(I) = \min\{i \in \mathbb{N} \mid \exists_k \text{ with } I \subseteq \{k+1, k+2, \dots, k+i\} \text{ modulo } (n)\}.$$

LEMMA 3.2. *Let I, J be disjoint subsets of N, and let $K \subseteq N$, with $I \cup J \subseteq K$. Let $X(i)$ be a restricted simple module $\neq S$ for each $i \in J$. Suppose that either*

(i) $|K \setminus (I \cup J)| > 0$ *and* $|K \setminus I| > 1$ (*e.g., if $I \cup J \subsetneq K$ and J is nonempty*) *or*

(ii) $\mathrm{diam}(K \setminus I) \geq N_0$. *Then*

$$\mathrm{Ext}^1_{FG(n)}\left(S_K, S_I \otimes \bigotimes_{i \in J} X(i)_i\right) = 0.$$

Proof. Let $k \in N \setminus K$. Untwisting k times, if necessary, and applying Lemma 2.3(a) if (i) holds, or Lemma 2.3(c) if (ii) holds, we see that S_K cannot be a composition factor of $S_k \otimes S_k \otimes S_I \otimes \bigotimes_{i \in J} X(i)_i$. The proof concludes by Lemma 3.1 and downward induction on $|K|$, as S_N is projective. ∎

LEMMA 3.3. *Let I, J be disjoint subsets of N, and let $K \subseteq N$. Let $X(i)$ be a restricted simple module $\neq S$ for each $i \in J$. If $|K \setminus (K \cap [I \cup J])| > 1$, then*

$$\mathrm{Ext}^1_{FG(n)}\left(S_K, S_I \otimes \bigotimes_{i \in J} X(i)_i\right) = 0.$$

Proof. We may assume $I \subseteq K$, as S is self-dual. Choose $k \in N \setminus K$. We observe that $S_k \otimes (S_I \otimes \bigotimes_{i \in J \cap (K \cup \{k\})} X(i)_i)$ has a quotient of the form $Y_k \otimes S_I \otimes \bigotimes_{i \in J \cap K} X(i)_i$, for some restricted simple module Y (possibly $Y = S$.) Furthermore, we claim that $S_K \otimes \bigotimes_{i \in J \setminus [J \cap (K \cup \{k\})]} X(i)_i^*$ cannot be a composition factor of $S_k \otimes Y_k \otimes S_I \otimes \bigotimes_{i \in J \cap K} X(i)_i$. This is shown by untwisting k times and applying Lemma 2.3(a), since $m_{p^n}(S \otimes Y) \le m_{p^n}(S) + m_{p^n}(Y) \le 2m(S)$, and because of the assumption on K. Therefore we have

$$\begin{aligned}
&\dim_F\Big(\operatorname{Ext}^1_{FG(n)}\Big(S_K, S_I \otimes \bigotimes_{i\in J} X(i)_i\Big)\Big)\\
&\quad = \dim_F\Big(\operatorname{Ext}^1_{FG(n)}\Big(S_K \otimes \bigotimes_{i\in J\setminus[J\cap(K\cup\{k\})]} X(i)_i^*,\\
&\qquad\qquad S_I \otimes \bigotimes_{i\in J\cap(K\cup\{k\})} X(i)_i\Big)\Big)\\
&\quad \le \dim_F\Big(\operatorname{Ext}^1_{FG(n)}\Big(S_{K\cup\{k\}} \otimes \bigotimes_{i\in J\setminus[J\cap(K\cup\{k\})]} X(i)_i^*,\\
&\qquad\qquad Y_k \otimes S_I \otimes \bigotimes_{i\in J\cap K} X(i)_i\Big)\Big)\\
&\quad = \dim_F\Big(\operatorname{Ext}^1_{FG(n)}\Big(S_{K\cup\{k\}}, Y_k \otimes S_I \otimes \bigotimes_{i\in J\setminus(J\cap\{k\})} X(i)_i\Big)\Big),
\end{aligned}$$

by Lemma 3.1. The result then follows by downward induction on $|K|$. ∎

Since S is self-dual, we then immediately obtain:

COROLLARY 3.4. *Let I, J be disjoint subsets of N with $|I| > 1$, then*

$$\operatorname{Ext}^1_{FG(n)}\Big(F, S_I \otimes \bigotimes_{i\in J} X(i)_i\Big) = 0,$$

whenever the $X(i)$ are restricted simple modules.

LEMMA 3.5. *Let J be a subset of N with* $\operatorname{diam}(J) > N_0$. *Let $X(i)$ be a restricted simple module $\notin \{F, S\}$ for each $i \in J$. Let K be an arbitrary subset of N. Then*

$$\operatorname{Ext}^1_{FG(n)}\Big(S_K, \bigotimes_{i\in J} X(i)_i\Big) = 0.$$

Proof. If $J \subseteq K$, we apply Lemma 3.2(ii). Thus, we may assume otherwise, let $k \in J \setminus (J \cap K)$ and find for each $i \in J$ a restricted simple module $Y(i) \notin \{F, S\}$ such that $\dim_F(\mathrm{Ext}^1_{FG(n)}(S_K, \bigotimes_{i \in J} X(i)_i)) \le \dim_F(\mathrm{Ext}^1_{FG(n)}(S_{K \cup \{k\}}, \bigotimes_{i \in J} Y(i)_i))$, iterating, if necessary, to reduce to the case $J \subseteq K$. To accomplish this, we use Lemma 3.1 to show that

$$\begin{aligned}
&\dim_F\left(\mathrm{Ext}^1_{FG(n)}\left(S_K, \bigotimes_{i \in J} X(i)_i\right)\right)\\
&\quad = \dim_F\left(\mathrm{Ext}^1_{FG(n)}\left(S_K \otimes \bigotimes_{i \in J \setminus [(J \cap K) \cup \{k\}]} X(i)^*_i,\right.\right.\\
&\qquad\qquad\qquad\qquad \left.\left. X(k)_k \otimes \bigotimes_{i \in J \cap K} X(i)_i\right)\right)\\
&\quad \le \dim_F\left(\mathrm{Ext}^1_{FG(n)}\left(S_{K \cup \{k\}} \otimes \bigotimes_{i \in J \setminus [(J \cap K) \cup \{k\}]} X(i)^*_i,\right.\right.\\
&\qquad\qquad\qquad\qquad \left.\left. B_k \otimes \bigotimes_{i \in J \cap K} X(i)_i\right)\right)\\
&\quad = \dim_F\left(\mathrm{Ext}^1_{FG(n)}\left(S_{K \cup \{k\}}, B_k \otimes \bigotimes_{i \in J \setminus \{k\}} X(i)_i\right)\right),
\end{aligned}$$

where $B \notin \{F, S\}$ is a restricted simple quotient of $S \otimes X(k)$ other than the restricted Steinberg module. The inequality follows from Lemma 3.1 because we can show that $S_K \otimes \bigotimes_{i \in J \setminus [(J \cap K) \cup \{k\}]} X(i)^*_i$ cannot be a composition factor of $(S_k \otimes B_k) \otimes \bigotimes_{i \in J \cap K} X(i)_i$ by applying Lemma 2.3(c) after untwisting k times and considering the assumption on J. ∎

4. CONCLUDING ARGUMENTS

Let I, J be disjoint subsets of N. Let $X(i)$ be a restricted simple module $\notin \{F, S\}$ for each $i \in J$. By untwisting, and using the lemmas in the preceding section, we have reduced the computation of

$$H^1\left(G(n), S_I \otimes \bigotimes_{i \in J} X(i)_i\right) := \mathrm{Ext}^1_{FG(n)}\left(F, S_I \otimes \bigotimes_{i \in J} X(i)_i\right),$$

to the cases where $J \subseteq \{0, 1, \ldots, N_0 - 1\}$ and $|I| \le 1$ (cf. Corollary 3.4 and Lemma 3.5). We wish to further reduce to $I = \varnothing$ whenever $J \ne \varnothing$.

Suppose that J is nonempty and that $I = \{k\}$. Let $J = \{i_1, \ldots, i_l\}$ with $i_1 < i_2 < \cdots < i_l$. Suppose that we have proven (using Lemma 3.1) the chain of inequalities,

$$\begin{aligned}
&\dim_F\left(\operatorname{Ext}^1_{FG(n)}\left(S_k, \bigotimes_{i\in J} X(i)_i\right)\right)\\
&\quad\le \dim_F\left(\operatorname{Ext}^1_{FG(n)}\left(S_k\otimes S_{i_1}, \tilde{X}(i_1)_{i_1}\otimes \bigotimes_{\substack{i\in J\\ i>i_1}} X(i)_i\right)\right)\\
&\quad\cdots \le \dim_F\left(\operatorname{Ext}^1_{FG(n)}\left(S_k\otimes S_{i_1}\otimes S_{i_2}\otimes\cdots\otimes S_{i_{t-1}},\right.\right.\\
&\qquad\qquad\left.\left.\bigotimes_{\substack{i\in J\\ i\le i_{t-1}}} \tilde{X}(i)_i\otimes \bigotimes_{\substack{i\in J\\ i>i_{t-1}}} X(i)_i\right)\right)
\end{aligned}$$

(where for each $i\in J$, $\tilde{X}(i)\notin\{F,S\}$ denotes a restricted simple quotient of $S\otimes X(i)$ other than the restricted Steinberg module). To continue the induction, we need to show that

$$\begin{aligned}
&\dim_F\left(\operatorname{Ext}^1_{FG(n)}\left(S_k\otimes \bigotimes_{\substack{i\in J\\ i\le i_{t-1}}} S_i, \bigotimes_{\substack{i\in J\\ i\le i_{t-1}}} \tilde{X}(i)_i\otimes \bigotimes_{\substack{i\in J\\ i>i_{t-1}}} X(i)_i\right)\right)\\
&\quad= \dim_F\left(\operatorname{Ext}^1_{FG(n)}\left(S_k\otimes \bigotimes_{\substack{i\in J\\ i\le i_{t-1}}} S_i\otimes \bigotimes_{\substack{i\in J\\ i>i_t}} X(i)^*_i, X(i_t)_{i_t}\otimes \bigotimes_{\substack{i\in J\\ i\le i_{t-1}}} \tilde{X}(i)_i\right)\right)\\
&\quad\le \dim_F\left(\operatorname{Ext}^1_{FG(n)}\left(S_k\otimes \bigotimes_{\substack{i\in J\\ i\le i_t}} S_i\otimes \bigotimes_{\substack{i\in J\\ i>i_t}} X(i)^*_i, \tilde{X}(i_t)_{i_t}\otimes \bigotimes_{\substack{i\in J\\ i\le i_{t-1}}} \tilde{X}(i)_i\right)\right).
\end{aligned}$$

To prove this, it suffices because of Lemma 3.1 to show that $S_k\otimes \bigotimes_{i\in J_{i\le i_{t-1}}} S_i\otimes \bigotimes_{i\in J_{i>i_t}} X(i)^*_i$ is not a composition factor of $S_{i_t}\otimes \tilde{X}(i_t)_{i_t} \bigotimes_{i\in J_{i\le i_{t-1}}} \tilde{X}(i)_i$. However, this follows by Lemma 2.3(b) after untwisting i_t times, since $m_{p^n}(S\otimes \tilde{X}(i_t))\le m_{p^n}(S)+m_{p^n}(\tilde{X}(i_t))<2m(S)$.

By induction, then, we have

$$\begin{aligned}
&\dim_F\left(\operatorname{Ext}^1_{FG(n)}\left(S_k, \bigotimes_{i\in J} X(i)_i\right)\right)\\
&\quad\cdots \le \dim_F\left(\operatorname{Ext}^1_{FG(n)}\left(S_{i_1}\otimes S_{i_2}\otimes\cdots\otimes S_{i_l}\otimes S_k, \bigotimes_{i\in J}\tilde{X}(i)_i\right)\right),
\end{aligned}$$

which is zero by Lemma 3.2(i).

We now define $N(G, p)$ using Cline, Parshall, Scott, and Van der Kallen's formula (cf. [4]). For a fixed prime p, and simply connected semisimple algebraic group G defined and split over the prime field $\mathbb{F}_p$, $N(G, p)$ will be defined as the bound N on n so that the restriction map of Theorem 7.1 of [4],

$$\operatorname{Ext}^1_G(F, L(\mu)_e) \to \operatorname{Ext}^1_{FG(n)}(F, L(\mu)_e)$$

is an isomorphism for all p^{N_0}-restricted weights μ for G, for all $n > N$, and for all sufficiently large e. We may define such a quantity since there are only a finite number of p^{N_0}-restricted weights. The argument is similar for the twisted groups (cf. [3]).

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